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Modelling dynamics in transportation networks: State of the art and future developments

Ennio Cascetta*
Department of Transportation Engineering, University of Napoli Federico II, via Claudio, 21, 80125 Napoli, Italy

Giulio Erberto Cantarella*
Institute of Civil Engineering, University of Reggio Calabria, via E. Cuzzocrea, 69, 89100 Reggio Calabria, Italy

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Abstract

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We present a general framework for the simulation of dynamics in transportation networks. Models and algorithms for both within-day and day-to-day dynamic traffic assignment are discussed. The proposed framework includes static models as a particular case, and is general enough to cover most of the existing models.

Keywords: Transportation networks; day-to-day dynamics; within-day dynamics; equilibrium analysis.

1. Introduction

Traffic assignment models are used to simulate link flows on transportation networks and the resulting link performances, such as travel times, congestion, pollution, and energy consumption. They are the basic tool for long-term and short-term planning and design of both urban and extra-urban transportation networks. Recently, on-line applications have also been proposed for supporting real-time control operations. The following results will mainly deal with road networks, but they can be quite easily extended to transit or railway networks.

Most traffic assignment models share a common structure made up by

a *demand model* simulating users' behaviour,

a *supply model* simulating the network performances,

a *supply/demand interaction model* simulating the interaction between users' behaviour and network performances.

In the following a general formal framework for dynamic assignment models, including static ones as a particular case, is decribed and several models proposed in the literature are classified by using this general common structure. In particular, after some general definitions and notations in Section 2, in Section 3 demand models are dealt with, then Section 4 describes

Correspondence to: E. Cascetta, Department of Transportation Engineering, University of Napoli Federico II, via Claudio, 21, 80125 Napoli, Italy.
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supply models, and Section 5 presents demand/supply interaction models. Some numerical examples are discussed in Section 6.

2. General definitions and notations

2.1. Day-to-day dynamics

Traditionally the demand/supply interaction has been studied and simulated following an equilibrium approach in which a self-reproducing or fixed-point state of the system is searched. This approach relies on elegant and well-developed mathematical foundations (at least for the within-day static case) which can be effectively solved for large-scale networks. Furthermore, it does not require an explicit modelling of users' memory and learning processes, since only the final state reached by the system is looked for, independently from the sequence of states needed to reach it. From this viewpoint the equilibrium approach could also be considered *day-to-day* or *inter-periodic static.*

However, the day-to-day static (or equilibrium) approach relies on some limitative assumptions from the theoretical point of view: mainly the existence, uniqueness and stability of the equilibrium state. Moreover, transients due to modifications of demand and/or supply and a statistical description of the state of the system, i.e. means, modes, moments and, more generally, frequency distributions of flows over time, cannot be simulated through equilibrium models. This implies that dynamic control strategies (such as adaptive traffic lights, variable message signs, route guidance systems, parking location choice support) reacting to perturbations in demand and/or supply cannot be effectively simulated through a day-to-day static or equilibrium approach.

Day-to-day or *inter-periodic dynamic* models can be seen as a generalisation of the equilibrium paradigm, allowing the removal of most of its limitative assumptions and a wider range of applications including the analysis of transients and of system convergence to different attractors (not necessarily equilibria or fixed points). From this viewpoint these models could be also called *disequilibrium models.*

Specification of day-to-day dynamic models, however, requires an explicit modelling of the system adjustment mechanism, including users' memory and learning processes and their interaction with operating control strategies. On the other hand, in this way the role of habit and non-compensatory behaviour in users' choice, needed to effectively assess dynamic control strategies, can be explicitly simulated.

Deterministic dynamic process models, based on the non-linear dynamic system theory, can be used to analyse the asymptotic behaviour of the system. They can also be adopted to obtain a consistent formulation of user equilibrium, which can be seen as a fixed-point attractor of a deterministic process, under some hypotheses on users' learning mechanisms and switching behaviour. Stochastic dynamic process models, based on stochastic process theory, allow an explicit simulation of the intrinsic randomness of both demand and supply.

2.2. Within-day dynamics

If travel demand is assumed to be (approximately) constant over a reference period (e.g. the morning peak period) which is large enough to allow the system to reach a stationary flow pattern, the assignment model is called *within-day* or *infra-periodic static.* Time-dependent demand (due for instance to the rush-hour) and/or changes in supply (due for instance to incidents or weather conditions) generally determine time-dependent flows and over-saturation queues, which can be only simulated through *within-day* or *infra-periodic dynamic* models. These models also allow to take into account the effects of real-time control strategies (such as variable message signs, radio broadcasting, etc.).

The extension of within-day static models to take into account within-day dynamics is by no means straightforward, since within-day dynamic supply modelling requires entirely new definitions of relevant variables and a reformulation of the problem, even though within-day dynamic users' behaviour can be modelled through an extension of the within-day static case.

Generally both disaggregate and aggregate approaches can be followed in order to develop demand and supply models (and their interaction). While disaggregate models can be more sophisticated and accurate, they show some disadvantages. Firstly they do not allow to draw general results about system properties (e.g. conditions assuring equilibrium stability). Furthermore, they are computationally demanding both in terms of memory requirements and computer running time, and give results with an unnecessarily high level of detail for many applications which require only aggregate performance parameters. On the whole, micro simulation models appear better suited as a second-step tool for the evaluation of detailed network performances for small areas of particular interest.

The alternative is an aggregate approach leading to macro-simulation models, where only the cumulative effects of users' choices are simulated. Both continuous and discrete-time aggregate approaches are in principle possible. The continuous-time approach is mainly of theoretical interest, as all practical solution methods rely on some kind of time discretization, possibly leading to inconsistencies between the model formulation and the computational algorithms used to solve it. In addition, following a continuous-time approach, consistent definitions of some flow variables and corresponding calibration experiments seem quite problematic from a practical point of view. In any case, the continuous-time approach can be seen as a limit case of the discrete-time approach. For the above reasons, an aggregate discrete-time approach seems better suited for a large number of practical applications and will be adopted in this paper.

2.3. General notations

An assignment model may be applied to time periods which may be only a part of the whole day, such as the morning peak hour; in the following, for the sake of simplicity, the reference period will be called day (denoted by index t). The sequence of "days" may be interpreted as successive similar periods, e.g. working-day morning peak periods, or as a sequence of fictitious time periods in which the system is observed. The latter case, however, can make the physical interpretation and empirical validation of the models difficult.

Users are assumed grouped into n_i classes, such that all the users of class i are homogeneous in all behavioural characteristics; in particular, they travel (possibly for the same purpose) between the same O–D pair, and have a common set K_i of available paths. It is assumed that each day t, d_i^t users of class i per time-unit travel between the corresponding O–D pair. Usually O–D flows are assumed constant over successive days, $d_i^t = d_i$. Elasticity of demand can be dealt with by introducing a fictitious alternative (i.e. a path for within-day static models or a departure interval for within-day dynamic models), corresponding to the choice of not moving at all; in this case d_i is the number of potential travellers for user class i. In the following let $\mathbf{d}$ be the $(n_i \times 1)$ demand vector. More generally, the transportation demand should be modelled as a random variable, with expected value given by $\mathbf{d}$.

In a within-day static model, the result of the users' behaviour at day t can be expressed by the flow on each path k, F_k^t. The network performances can be expressed by the generalised transportation cost on each path k, C_k^t. In a within-day dynamic model, according to a discrete-time approach, the reference period is assumed to be divided into n_j intervals which are assumed to be, with no loss of generality, of equal length T, and F_{jk}^t is the number of users per unit of time, or the path flow, leaving during interval j and entering path k, at day t (where k is the path actually followed until the final destination, taking into account en-route diversions, if any). The corresponding path cost experienced by users is C^t_{jk}. In both cases let $\mathbf{F}^t$ be the $(n_{jk} \times 1)$ path flow vector at day t, and $\mathbf{C}^t$ be the $(n_{jk} \times 1)$ path cost vector at day t, where $n_{jk} = n_j$ or $n_{jk} = n_j \cdot n_k$ for within-day static or dynamic models, respectively.

The *demand model* simulating users' behaviour expresses how path costs affect path flows (and more generally demand elasticity), while the *supply model* simulating the network performances expresses the relationship between path costs and flows in a given day.

3. Demand models

In the following a general framework for within-day dynamic users' behaviour modelling is presented, together with some simplified formulations proposed in the literature more suitable for practical applications. Modelling of both users' learning and forecasting processes (Section 3.1) and choice behaviour (Section 3.2) are dealt with. In addition, the relationship between the proposed framework and the strategies for the management of information to users is addressed (Section 3.3). Within-day static models are not explicitly dealt with since they turn out to be a simplification of the within-day dynamic ones.

3.1. Learning and forecasting processes

Users do not know in advance the actual costs they will experience during their trip. Thus it is assumed that they make their choices according to forecasted (or perceived) path costs, resulting from their memory and learning processes. Personal experience is usually complemented by information exchanged with other users and possibly provided by an information system.

Generally forecasted costs depend on costs having occurred in previous days. Furthermore, if some users have access to a real-time informative system, their forecasted costs may depend also on costs occurring during the current day. Similar conditions occur if an adaptive route choice behaviour is assumed for users. For the sake of simplicity in this section forecasted costs are assumed depending only on costs having occurred in previous days.

Learning and forecasting processes can be modelled through filters applied to path costs having occurred in previous days. Let Y^t_{jk} be the forecasted (generalised transportation) cost relative to path k and departure interval j (averaged among all the users of class i, $k \in K_i$), and let $\mathbf{Y}^t$ be the $(n_{jk} \times 1)$ forecasted path cost vector at day t.

Two types of filter can be considered, both avoiding infinite-dimensional representation of the system state:

$$\text{type a} \quad \mathbf{Y}^t = Y(\mathbf{C}^{t-1}, \mathbf{Y}^{t-1}), \tag{3.1a}$$

$$\text{type b} \quad \mathbf{Y}^t = Y(\mathbf{C}^{t-h}, h = 1, \ldots, m). \tag{3.1b}$$

Other types of filters are built up by mixing together these two types, or by applying them to the forecasting error, $(\mathbf{Y}^{t-1} - \mathbf{C}^{t-1})$. The results presented in the following can be easily transferred to these other types.

The distribution of perceived path costs across users can be obtained following an aggregate approach (aggregated memory) in which individual differences are taken into account through random residuals with respect to the average forecasted path costs in the choice probability model (Section 3.2). On the other hand, the above described filters could be used to model the memory of each individual user by weighting differently the information relative to experienced and non-experienced path costs (disaggregate memory).

Learning and forecasting filters are called time-homogeneous if equal values of the path costs in previous days produce the same value of the filter whatever the day. Filters which are both time-homogeneous and linear are called conservative,

$$\mathbf{Y}^t = \mathbf{R}\cdot\mathbf{C}^{t-1} + (\mathbf{I}-\mathbf{R})\cdot\mathbf{Y}^{t-1}, \tag{3.2a}$$

$$\mathbf{Y}^t = \sum_{h=1}^{m} \mathbf{R}_h\cdot\mathbf{C}^{t-h} \quad \text{with} \quad \sum_{h=1}^{m} \mathbf{R}_h = \mathbf{I}, \tag{3.2b}$$

where $\mathbf{R}_h$ and $\mathbf{R}$ are duly defined diagonal matrices with elements in the range [0, 1].

Although there is still limited empirical evidence on forecasting processes, the filters most used in the literature [11, 22, 26, 27] can be cast in one of the following two classes:

- exponential smoothing (type a),

$$\mathbf{Y}^t = \beta\cdot\mathbf{C}^{t-1} + (1-\beta)\cdot\mathbf{Y}^{t-1} \quad \text{with} \quad \beta\in[0,1], \tag{3.3a}$$

- weighted average (type b),

$$\mathbf{Y}^t = \sum_{h=1}^{m} \beta_h\mathbf{C}^{t-h} \quad \text{with} \quad \sum_{h=1}^{m} \beta_h = 1 \text{ and } \beta_h \geqslant 0. \tag{3.3b}$$

If $m=1$ and $\beta_1 = 1$ in the weighted average or $\beta = 1$ in the exponential smoothing filter, only yesterday's experience influences daily path choices. From the practical point of view the two types of forecasting filters are almost equivalent, since the exponential filter can also be expressed in the weighted average form with only a limited number of previous days costs, discarding remote days with small weights.

Iida et al. [38, 39] proposed a slightly different filter based on forecasting error:

$$\mathbf{E}^t = \sum_{h=1}^{m} \beta_h(\mathbf{Y}^{t-h} - \mathbf{C}^{t-h}) + \beta_0 \quad \text{with} \quad \beta_h \geqslant 0,$$

$$\mathbf{Y}^t = \mathbf{C}^{t-1} + \mathbf{E}^t.$$

3.2. Users' choice behaviour

In a within-day dynamic system users' choices are basically related to physical and/or fictitious paths representing other choice dimensions such as making the trip or not, mode, etc. The fraction of users of class i starting at interval j and following path $k\in K_i$ at day t should be modelled through random variables with expected values given by choice probabilities.

Let $x^t_{i,jk}$ be the choice probability of the departure-interval–path pair (j, k) for a user of class i (equal to zero if $k \notin K_i$) and let $\mathbf{X}^t$ be the $(n_{jk}\times n_i)$ choice probability matrix.

The (expected value of the) path flow vector can be expressed as

$$F^t_{jk} = x^t_{i,jk}\cdot d_i \quad k\in K_i,$$
$$\mathbf{F}^t = \mathbf{X}^t\cdot\mathbf{d}. \tag{3.4}$$

Choice probabilities can be modelled to explicitly take into account whether users reconsider their previous day's choices or confirm them, that is, the choice switching process.

The probability that a user of class i chooses departure interval j and path k at day t generally depends on the pair (j', k') chosen the previous day, either due to habit and conservative behaviour or to a desire for variety. Let $p^t_{i,jk/j'k'}$ be the (transaction) probability that a user of class i chooses interval j and path k at day t, given that j' and k' are the interval and the path chosen the previous day $t-1$, and let $\mathbf{T}^t$ be the $(n_{jk} \times n_{jk})$ transaction matrix. Then we have

$$x^t_{i,jk} = \sum_{j'k'} x^{t-1}_{i,j'k'} \cdot p^t_{i,jk/j'k'},$$

$$\mathbf{X}^t = \mathbf{T}^t \cdot \mathbf{X}^{t-1}.$$

Since it is assumed that users make their choice according to forecasted path costs, it generally follows that

$$\mathbf{T}^t = T(\mathbf{Y}^t).$$

Hence the evolution over time of the path flow vector is generally defined by

$$\mathbf{F}^t = T(\mathbf{Y}^t) \cdot \mathbf{F}^{t-1}. \tag{3.5}$$

Let $p^t_{i,c/j'k'}$ be the probability that at day t a user of class i reconsiders (but not necessarily changes) interval j' and path k' chosen the previous day, and $p^t_{i,jk/c,j'k'}$ be the probability that user of class i reconsidering the interval j' and path k' chosen the previous day $t-1$, chooses interval j and path k at day t. Then the transition probability can be expressed as

$$p^t_{i,jk/j'k'} = p^t_{i,jk/c,j'k'} \cdot p^t_{i,c/j'k'} \qquad \text{if } j',k' \neq j,k,$$

$$p^t_{i,jk/jk} = (1 - p^t_{i,c/jk}) + p^t_{i,jk/c,jk} \cdot p^t_{i,c/jk} \quad \text{if } j',k' = j,k.$$

According to this formulation the choice probability can be specified as

$$x^t_{i,jk} = \left(\sum_{j'=1}^{nj} \sum_{k' \in K_i} (p^t_{i,jk/c,j'k'} \cdot p^t_{i,c/j'k'} \cdot x^{t-1}_{i,j'k'}) \right) + ((1 - p^t_{i,c/jk}) \cdot x^{t-1}_{i,jk})$$

If it is assumed that the choice probabilities do not depend on the choice made the previous day, a simpler model is obtained. Let $p^t_{i,c}$ be the probability of reconsidering the previous day's choice for a user of class i, and $p^t_{i,jk/c}$ be the probability of choosing interval j and path k for a user reconsidering the previous day's choice. Then it follows that

$$p^t_{i,c/jk} = p^t_{i,c}, \qquad p^t_{i,jk/c,j'k'} = p^t_{i,jk/c}$$

and the following simplified formulation is obtained:

$$x^t_{i,jk} = p^t_{i,jk/c} \cdot p^t_{i,c} + (1 - p^t_{i,c}) \cdot x^{t-1}_{i,jk}.$$

Thus let $\mathbf{Q}^t$ be the $(n_{jk} \times n_{jk})$ switching matrix, a diagonal matrix, with the (j, k) element on the main diagonal equal to the probability of reconsidering the previous day's choice $p^t_{i,c}$ with $k \in K_i$; and let $\mathbf{P}^t$ be the $(n_{jk} \times n_i)$ choice probability matrix, with element $p^t_{i,jk/c}$. Then we have

$$\mathbf{X}^t = \mathbf{Q}^t \cdot \mathbf{P}^t + (\mathbf{I} - \mathbf{Q}^t) \cdot \mathbf{X}^{t-1}.$$

Moreover, since the choice probabilities depend on the forecasted path costs it follows that

$$\mathbf{P}^t = P(\mathbf{Y}^t) \quad \text{and} \quad \mathbf{Q}^t = Q(\mathbf{Y}^t).$$

Hence, the path flow vector at day t is expressed by

$$\mathbf{F}^t = Q(\mathbf{Y}^t) \cdot P(\mathbf{Y}^t) \cdot \mathbf{d} + (\mathbf{I} - Q(\mathbf{Y}^t)) \cdot \mathbf{F}^{t-1}. \tag{3.6}$$

According to this simplified approach (referred to as the QP demand model in the following), the probability to choose a given departure time and path pair is defined as the result of the probability to reconsider the previous day's choice (*switching choice behaviour*), expressed by matrix $\mathbf{Q}$, and the conditional probability of choosing that path and departure time pair given that the previous day's choice is reconsidered (*departure time and path choice behaviour*) expressed by matrix **P**. Most models existing in the literature follow this approach.

Switching choice behaviour, that is, matrix **Q**, can be modelled using different approaches. The simplest and more frequently adopted one assumes that each day t users reconsider (but not necessarily change) their previous day's choices with a constant probability α not depending on day t [2, 3, 14]:

$$\mathbf{Q} = \alpha \mathbf{I}.$$

Then it turns out that

$$\mathbf{F}^t = \alpha \cdot \mathbf{P}^t \cdot \mathbf{d} + (1 - \alpha) \cdot \mathbf{F}^{t-1}. \tag{3.7}$$

More sophisticated models of path switching can be adopted according to the more general formulations presented above, as for instance an extra utility for the alternative chosen the previous day, for aggregated memory [14], or a (deterministic or stochastic) threshold for the difference between the forecasted and the actual cost of the alternative chosen the previous day, for disaggregate memory [26, 27].

The probability to choose departure interval j and path k for a user of class i, reconsidering the previous day's choice at day t, $p^t_{i,jk/c}$, that is, an element of matrix **P**, is usually modelled through random utility discrete choice models [4, 5, 31]. It is assumed that at day t each user of class i associates to a departure interval and path pair (j, k) a perceived disutility $U^t_{i,jk}$ which is considered as a random variable distributed across the population of users. Perceived disutility $U^t_{i,jk}$ is expressed as the sum of a systematic term, called average perceived (or forecasted according to the terminology adopted in this paper) disutility $V^t_{i,jk}$, and a random residual $\varepsilon^t_{i,jk}$, which models the perception (and modelling) errors.

The average perceived disutility $V^t_{i,jk}$ can be defined as a linear combination of the forecasted generalised transportation path cost Y^t_{jk} and of the scheduled delay determined from the chosen departure interval j, the desired arrival time, and the forecasted path travel time [2, 3, 18, 32]. The forecasted path travel time, if different from the forecasted generalised path cost, can be computed through filters, similar to those described in Section 3.1.

The explicit expression of the choice probabilities depends on the random distribution of residuals. The Weibull–Gumble distribution leads to the well-known Multinomial Logit model,

$$p^t_{i,jk/c} = \exp(-\theta V^t_{i,jk}) \Big/ \sum_{m,n} \exp(-\theta V^t_{i,mn}),$$

where $\theta = \sqrt{6}\pi/\sigma$ and σ is the standard deviation of random residuals. The MultiVariate Normal distribution leads to the Probit model (which cannot be described by a closed formula), defined through a complete variance–covariance matrix Σ between random residuals.

In a simplified approach to model users' behaviour, usually referred to as "deterministic behaviour", the perceived disutility $\mathbf{U}^t$ is assumed equal to the systematic term $\mathbf{V}^t$, thus assuming that random residuals are zero, and only alternatives with minimum disutility are chosen. By comparison the most general approach to users' behaviour modelling, based on random utility theory, is called "stochastic behaviour". These definitions should be not confused with deterministic or stochastic process models used for simulating demand/supply interaction.

Nested models of departure interval and path choice are often adopted,

$$p^t_{i,jk/c} = p^t_{i,j/c} \times p^t_{i,k/jc},$$

where $p^t_{i,j/c}$ is the probability to choose departure interval j for a user of class i (reconsidering the previous day's choice) at day t, and $p^t_{i,k/jc}$ is the probability to choose path k for a user of class i (reconsidering the previous day's choice) who has chosen departure interval j at day t. Both these probabilities can be specified through random utility models.

3.3. Types of users' behaviour and informative systems

Users moving in a transportation network are generally assumed to make a number of choices at their origin (destination, mode, departure time, path and, possibly, parking type and location). After leaving the origin, while-trip re-routing may occur (possibly at each node of the network) according to the users' adaptive behaviour, hence the path (and other choices) actually used may differ from the initially chosen one.

Moreover, an informative system may supply messages (information and/or indications) to users before they leave (pre-trip) and/or while they are travelling (en-route). The informative system may define the messages to users by using information only about network conditions on previous days (static or historical) or by combining it with information about present network conditions (dynamic or real-time), and it may give messages before the departure time (pre-trip informative systems) or the arrival time (en-route informative systems). Different classes of users with different types of available information may be present at the same time on the network.

Pre-trip choice of departure time and initial path, which depends on forecasted path costs based on information available until the departure interval, can be simulated through random utility models, as described in the previous section, with advised users' behaviour modelled through a smaller standard deviation of perception errors. A nested formulation with a slight modification is needed in the case of a real-time pre-trip informative system [16], since in this case users' forecasted path costs for a given departure interval may vary within-day due to new information (at least for informed users), as may departure interval choice probabilities.

En-route definition of the path actually used takes into account while-trip re-routing due to local traffic conditions and/or in response to any source of on-board information, such as those provided by radio broadcast, variable message signs, route guidance systems. It depends on forecasted path costs based on information available en-route and it is simulated only for advised users.

En-route behaviour has not yet received great attention in the literature. However, if advised users receive information their behaviour is generally modelled through random utility models with reduced error variances. On the other hand, if they receive indications their behaviour is modelled through a compliance fraction, i.e the fraction of users which will follow system guidance.

It should be stressed that while-trip re-routing greatly affects also the supply model, as it will be shown in the following section.

4. Supply models

The network topology is generally modelled through a graph with n_k paths and n_l links. Let $\mathbf{A}$ be the $(n_l \times n_k)$ link-path incidence matrix ($\mathbf{A}'$ is its transpose), with $a_{lk} = 1$ if link l belongs to path k and $a_{lk} = 0$ otherwise.

In the within-day static case the supply model can easily be formulated through the link-path incidence matrix, while in the within-day dynamic case a more complex approach is needed. Therefore the two cases are described separately in the following.

4.1. Within-day static supply models

Let f_l^t be the flow on link l (that is, the average number of users crossing in a time-unit each cross-section of the link) and let $\mathbf{f}^t$ be the $(n_l \times 1)$ link flow vector at day t. The *Network Loading*

(NL) map, which expresses the relationship between link and path flows, is given by

$$f_l^t = \sum_k a_{lk} F_k^t, \qquad (4.1)$$
$$\mathbf{f}^t = \mathbf{A} \cdot \mathbf{F}^t.$$

Let c_l^t be the generalised transportation cost on link l at day t, and let $\mathbf{c}^t$ be the $(n_l \times 1)$ link cost vectors at day t. It is generally assumed that link costs are deterministic functions of the link flow vector (even though also stochastic models can fit in the proposed framework):

$$c_l^t = c_l(\mathbf{f}^t), \qquad (4.2)$$
$$\mathbf{c}^t = c(\mathbf{f}^t).$$

According to notations in Section 2, C_k^t denotes the cost on path k, and $\mathbf{C}^t$ the corresponding path cost vector, therefore it follows that

$$C_k^t = \sum_l a_{lk} c_l, \qquad (4.3)$$
$$\mathbf{C}^t = \mathbf{A}' \cdot \mathbf{c}^t.$$

Hence, a within-day static supply model is completely specified once the link-path incidence matrix, which defines the network topology, and the link cost-flow functions are given, and can be expressed by combining (4.1), (4.2) and (4.3) into

$$\mathbf{C}^t = \mathbf{A}' \cdot c(\mathbf{A} \cdot \mathbf{F}^t). \qquad (4.4)$$

4.2. Within-day dynamic supply models

In an aggregate approach (average) link performances are related to a variable associated to each link and to each time interval, usually denoted by "link flow". In a within-day dynamic network the definition of link flow is not unique as in the static case since time and space averages in general do not coincide. Different definitions can be adopted corresponding to different modelling approaches, examples include average in-flow in a given cross-section or space–time average flow. For the time being, let f_{lh}^t be the flow, however defined, on link l during interval h, and let $\mathbf{f}^t$ be the $(n_j \cdot n_l \times 1)$ link flow vector at day t. The Dynamic Network Loading (DNL) map, that is, the relationship between link and path flows, can be formally expressed by

$$\mathbf{f}^t = \Phi[\mathbf{F}^t]. \qquad (4.5)$$

This map is generally non-linear, and cannot be defined analytically as in the within-day static case (4.1).

In the within-day dynamic case the time needed to traverse a link depends both on link performance characteristics, such as running speed or queue length, and the arrival time at the entrance node of the link, and the same condition occurs for the link cost, which cannot be defined through a function of only (the link and) the link flow vector as in the within-day static case. Hence, let $\mathbf{r}$ be the $(n_j \cdot n_l \times 1)$ link performance vector. In congested networks it is generally a function of the link flow vector,

$$\mathbf{r}^t = r(\mathbf{f}^t). \qquad (4.6)$$

Once the link flows are known through the DNL map, the link performance vector can be determined, and the arrival times can be sequentially computed link by link along each path for any given departure time. Then the corresponding link costs can be defined as functions of both link performances and arrival times. Hence, according to notations introduced in Section 2, let C^t_{jk} be the cost on path k averaged across all users leaving during interval j of day t and following path k, and let $\mathbf{C}^t$ be the corresponding path cost vector. Path costs for a given departure interval j can be computed from the corresponding link costs; then path costs turn out to be functions of the link performance vector,

$$\mathbf{C}^t = \Gamma[\mathbf{r}^t]. \tag{4.7}$$

A within-day dynamic supply model is obtained once both the DNL maps and the link performance-flow functions are specified,

$$\mathbf{C}^t = \Gamma[r(\Phi[\mathbf{F}^t])]. \tag{4.8}$$

4.3. Dynamic network loading

In this subsection the DNL map is addressed in more detail. A straightforward generalisation of within-day static formulations is not possible since, as said before, the definition of link flow is not unique, therefore link and path flows and costs must be consistently redefined. In addition, since the time-dependent link flows depend on the time-dependent link performance characteristics, which are generally functions of the link flows themselves, a fixed-point problem arises to define the DNL map, as described in the following. (The superscript t will be omitted for simplicity's sake.)

Let $b^{jk}_{lh} \in [0, 1]$ be the fraction of path flow F_{jk} contributing to link flow f_{lh} named *crossing fraction*. Then the flow on link l can be expressed as

$$f_{lh} = \sum_{j=1}^{h} \sum_{k} b^{jk}_{lh} F_{jk}.$$

If link l does not belong to path k, or the departure interval j of path flow F_{jk} is subsequent to interval h ($j > h$), or path flow F_{jk} is not on link l during interval h, the fraction b^{jk}_{lh} is equal to zero. Let $\mathbf{B}$ be the low triangular block $(n_l \cdot n_j \times n_k \cdot n_j)$ crossing fraction matrix, Then we have

$$\mathbf{f} = \mathbf{B} \cdot \mathbf{F}.$$

From this expression the crossing fraction matrix can be seen as a generalisation of the link-path incidence matrix $\mathbf{A}$, previously defined (for more details see [16]).

Crossing fractions b^{jk}_{lh}, consistent with the definition adopted for link flows, depend on the network topology and the time needed to reach link l travelling on path k and leaving during interval j. Therefore, they depend on travel times on links preceding link l on path k, which are functions of the link performance vector $\mathbf{r}$; hence

$$\mathbf{B} = B[\mathbf{r}].$$

Therefore, generally the relationship between link and path flows can be expressed as

$$\mathbf{f} = B[\mathbf{r}] \cdot \mathbf{F}. \tag{4.9}$$

On the other hand, as stated above, link performance characteristics in congested networks are generally functions of the link flows; therefore the following fixed-point problem arises, combining (4.6) and (4.9), in order to define the DNL map (4.5) between link and path flows:

$$\mathbf{f}^* = B[r(\mathbf{f}^*)] \cdot \mathbf{F}. \tag{4.10}$$

Once link performance functions (4.6) have been specified a dynamic network loading method essentially reduces to the method for computing crossing fractions, that is, the relationship $b^{jk}_{lh} = b^{jk}_{lh}[\mathbf{f}]$ needed to solve the fixed-point problem (4.10). It is worth noting that in non-congested networks where travel times are constant and independent of link flows, the link-path crossing matrix does not depend on link flows and the fixed-point problem reduces to

$$\mathbf{f}^* = \mathbf{B} \cdot \mathbf{F},$$

formally analogous with the within-day static case.

Several approaches have been proposed for the solution of the DNL, and are briefly summarised in the following. Some requirements that a fully satisfactory DNL method should meet are presented by Cascetta and Cantarella [16], while a comprehensive analysis of the state-of-art is reported by Di Gangi [19], who also describes an effective method for dealing with DNL.

Most of the existing methods do not clearly define link flows and/or can hardly be extended to include en-route diversions from the initially chosen path, because of the computational burden of keeping the identity of diverted path flows. Moreover, some of them rely on assumptions which do not necessarily rule out overtaking between users following the same path with different departure times. Among these, heuristic generalisation of within-day static methods have been proposed by Hammerslag [21], and by Janson [23]; exit-function methods have been firstly proposed by Merchant and Nemhauser [29], and then adopted by Carey [8, 9], Friesz et al. [20], Wie et al. [35], Papageorgiou [30], and Boyce et al. [37]; discretisation of a differential equation link flow model has been used by Vythoulkas [34], who extended the method previously proposed by Ben Akiva et al. [2, 3] for parallel path networks.

Other methods, more suitable for within-day dynamic applications, are based on a packet-approach in which users following the same path and entering the network during the same interval are grouped and moved as a single unit experiencing the same trip of the leader. Under these assumptions these methods can be effectively used to address DNL (CORQ for corridor networks proposed by Yagar [36]; CONTRAM for "deterministic" users' behaviour proposed by Leonard and Gower [24, 25]; the method proposed by Cascetta and Cantarella [14] for explicitly solving the fixed-point problem). Recently a generalisation of the grouped packet-approach to continuously spread packets has been proposed by Di Gangi [19].

5. Demand/supply interaction and day-to-day dynamics

The demand/supply interaction is simulated through a relationship between the previously described demand and supply models. As stated in the Introduction, different approaches can be adopted and will be described in the following together with their mutual relationship.

5.1. Deterministic dynamic models

Deterministic day-to-day dynamic models ignore random fluctuations of demand and/or supply. Therefore, the fraction of users of class i travelling on path k belonging to set K_i (and departing during interval j in within-day dynamic models) at day t is assumed equal to its average value, i.e. the choice probability, and the path cost vector is equal to the average path cost vector.

Generally the description of the *state of the system* depends on the type of learning and forecasting filter $Y(-)$. In the case of type-a filters (equation (3.1a)) the system state is completely described by the path flow and average forecasted cost vectors $(\mathbf{F}^t, \mathbf{Y}^t)$. On the other hand, if type-b filters (equation (3.1b)) are adopted, the system state is described by path flows in the previous m days, $(\mathbf{F}^t, \mathbf{F}^{t-h}, h = 1, \ldots, m)$, since forecasted path costs are deterministic functions of path costs in m previous days, and therefore of link or path flows in those days.

Many authors analysed the system dynamics through a deterministic process model. However, most of them either referred to particularly simple networks [6, 22] and/or memory and learning processes [1, 33]. General deterministic day-to-day dynamic models have been dealt with by Cascetta and Cantarella [15, 17].

In the following deterministic day-to-day dynamic models based on non-linear dynamic system theory are presented. Within-day dynamic models are firstly described, then the static ones, which result in special cases of the former ones.

5.1.1. Within-day dynamic models

A deterministic process model is expressed by recursive equations; in the most general approach we have

$$\mathbf{Y}^t = Y(\mathbf{Y}^{t-1}, \mathbf{C}^{t-1}) \quad \text{or} \quad \mathbf{Y}^t = Y(\mathbf{C}^{t-h}, h = 1, \ldots, m), \tag{5.1}$$

$$\mathbf{F}^t = T(\mathbf{Y}^t)\mathbf{F}^{t-1}, \tag{5.2}$$

with

$$\mathbf{C}^t = \Gamma[r(\Phi[\mathbf{F}^t])]. \tag{5.3}$$

Using the QP switching behaviour model (3.6), it turns out that

$$\mathbf{F}^t = Q(\mathbf{Y}^t)P(\mathbf{Y}^t)\mathbf{d} + (\mathbf{I} - Q(\mathbf{Y}^t))\mathbf{F}^{t-1}.$$

Moreover, instead of general filters, conservative filters (equation (3.2)) can be adopted,

$$\mathbf{Y}^t = \mathbf{R}\mathbf{C}^{t-1} + (\mathbf{I} - \mathbf{R})\mathbf{Y}^{t-1}$$

or

$$\mathbf{Y}^t = \sum_{h=1}^{m} \mathbf{R}_h \mathbf{C}^{t-h} \quad \text{with} \quad \sum_{h=1}^{m} \mathbf{R}_h = \mathbf{I},$$

where $\mathbf{R}_h$ and $\mathbf{R}$ are duly defined diagonal matrices with elements in the range [0, 1].

Once a starting state $(\mathbf{F}_0, \mathbf{Y}_0)$ is given, the deterministic dynamic process model can be easily solved by repeatedly applying the recursive equations until an attractor (if any) is reached. The *control parameters* are the parameters of forecasted cost filter $Y(-)$, of link performance functions $r(-)$, and of users' behaviour model, say $T(-)$ or $Q(-)$ and $P(-)$. If it is assumed that the probabilities of reconsidering the previous day's choice are strictly positive, $p^t_{i,c} > 0$, matrix $\mathbf{Q} = Q(-)$ is not singular. Similarly matrices $\mathbf{R}$ or $\mathbf{R}_h$ used for filter definition are assumed non-singular. Only in this case the model will show a real evolution over time, that is, the state may change from the starting one.

Generally the system may converge to different attractors, which can at least theoretically be identified by techniques from non-linear dynamic systems theory. Cascetta and Cantarella [17] report some comments on the conditions under which the system converges to an attractor, and about the different types of attractors which can be reached.

In the following fixed-point attractors, or self-reproducing states, are dealt with. They are defined by the following conditions:

$$\mathbf{Y}^t = \mathbf{Y}^{t-1} = \mathbf{Y}^*, \qquad \mathbf{F}^t = \mathbf{F}^{t-1} = \mathbf{F}^*.$$

According to the general formulation expressed by (5.1), (5.2) and (5.3) a fixed-point attractor is reached when the following conditions hold:

$$\mathbf{Y}^* = Y(\mathbf{C}^*, \mathbf{Y}^*), \tag{5.4}$$

$$|T(\mathbf{Y}^*) - \mathbf{I}| = 0, \tag{5.5}$$

$$\mathbf{C}^* = \Gamma[r(\Phi[\mathbf{F}^*])]. \tag{5.6}$$

With the QP formulation (equation (3.6) instead of (5.2)), a fixed-point attractor is given by (5.4), (5.6) and

$$\mathbf{F}^* = P(\mathbf{Y}^*) \cdot \mathbf{d}. \tag{5.7}$$

Moreover, using conservative filters, expressed by (3.2), equation (5.4) becomes (having assumed that matrix $\mathbf{R}$ is not singular)

$$\mathbf{Y}^* = \mathbf{C}^*, \tag{5.8}$$

thus leading to

$$\mathbf{F}^* = P(\Gamma[r(\Phi[\mathbf{F}^*])]) \cdot \mathbf{d} \tag{5.9}$$

with path costs given by (5.6)

5.1.2. Within-day static models

Similar formulations can also be adopted for within-day static models (changing accordingly the meaning of path and link cost and flow vectors), leading in the most general case to

$$\mathbf{Y}^t = Y(\mathbf{Y}^{t-1}, \mathbf{C}^{t-1}) \quad \text{or} \quad \mathbf{Y}^t = Y(\mathbf{C}^{t-h}, h = 1, \ldots, m), \tag{5.10}$$

$$\mathbf{F}^t = T(\mathbf{Y}^t)\mathbf{F}^{t-1}, \tag{5.11}$$

$$\mathbf{C}^t = \mathbf{A}'c(\mathbf{f}^t) = \mathbf{A}'c(\mathbf{AF}^t). \tag{5.12}$$

A QP approach for switching behaviour can be followed (equation (3.6) instead of (5.11)), and/or conservative filters can be adopted (equation (3.2) instead of (5.10)). In this case a link formulation can also be adopted,

$$\mathbf{y}^t = \mathbf{R} \cdot \mathbf{c}^{t-1} + (\mathbf{I} - \mathbf{R}) \cdot \mathbf{y}^{t-1} \tag{5.13a}$$

or

$$\mathbf{y}^t = \sum_{h=1}^{m} \mathbf{R}_h \cdot \mathbf{c}^{t-h} \quad \text{with} \quad \sum_{h=1}^{m} \mathbf{R}_h = \mathbf{I}, \tag{5.13b}$$

$$\mathbf{f}^t = Q(\mathbf{A}' \cdot \mathbf{y}^t) \cdot \mathbf{A} \cdot P(\mathbf{A}' \cdot \mathbf{y}^t) \cdot \mathbf{d} + (I - Q(\mathbf{A}' \cdot \mathbf{y}^t)) \cdot \mathbf{A} \cdot \mathbf{f}^{t-1}, \tag{5.14}$$

$$\mathbf{c}^t = c(\mathbf{f}^t), \tag{5.15}$$

with clear meaning of all the involved vectors.

Fixed-points, or self-reproducing states, can be defined as in the within-day dynamic case, leading to the following expression, according to the general formulation expressed by (5.10), (5.11) and (5.12):

$$\mathbf{Y}^* = Y(\mathbf{C}^*, \mathbf{Y}^*), \tag{5.16}$$

$$|T(\mathbf{Y}^*) - \mathbf{I}| = 0, \tag{5.17}$$

$$\mathbf{C}^* = \mathbf{A}' \cdot c(\mathbf{A} \cdot \mathbf{F}^*). \tag{5.18}$$

With the QP switching behaviour model (equation (3.6) instead of (5.11)), and using conservative filters (equation (3.2) instead of (5.16)) a fixed-point attractor is given by

$$\mathbf{F}^* = P(\mathbf{A}' \cdot c(\mathbf{A} \cdot \mathbf{F}^*)) \cdot \mathbf{d} \tag{5.19}$$

with $\mathbf{Y}^* = \mathbf{C}^* = \mathbf{A}' \cdot c(\mathbf{A} \cdot \mathbf{F}^*)$. In this case also a link formulation is possible:

$$\mathbf{f}^* = \mathbf{A} \cdot P(\mathbf{A}' \cdot c(\mathbf{f}^*)) \cdot \mathbf{d}, \tag{5.20}$$

with

$$\mathbf{y}^* = \mathbf{c}^* = c(\mathbf{f}^*).$$

5.2. Equilibrium (or day-to-day static) models

The modelling of demand/supply interactions has been traditionally addressed following an equilibrium approach in which a self-reproducing or fixed-point state of the system is searched. In other words, not only the path flows resulting from the demand model are assumed equal to the path flows which define the supply model, but also the forecasted costs, which influence users' behaviour, are assumed equal to actual costs,

$$\mathbf{Y}^t = \mathbf{C}^t.$$

It should be noted that an equilibrium state does not necessarily coincide with a fixed-point attractor of a deterministic dynamic process model.

This approach combined with the within-day static assumption yields the well-know user equilibrium models (among many others, reviews are in [7, 12, 31]):

$$\mathbf{F}^* = P(\mathbf{C}^*) \cdot \mathbf{d}, \tag{5.21}$$

$$\mathbf{C}^* = \mathbf{A}' c(\mathbf{A} \cdot \mathbf{F}^*), \tag{5.22}$$

or

$$\mathbf{F}^* = P(\mathbf{A}' c(\mathbf{A} \cdot \mathbf{F}^*)) \cdot \mathbf{d}, \tag{5.23}$$

which can be also stated in a link-flow formulation as

$$\mathbf{f}^* = \mathbf{A} \cdot P(\mathbf{A}' \cdot \mathbf{c}^*) \cdot \mathbf{d}, \tag{5.24}$$

$$\mathbf{c}^* = c(\mathbf{f}), \tag{5.25}$$

or

$$\mathbf{f}^* = \mathbf{A} \cdot P(\mathbf{A}' \cdot c(\mathbf{f}^*)) \cdot \mathbf{d}. \tag{5.26}$$

The case of "deterministic" users' behaviour, described in Section 3.2, can be seen as a limit case of this formulation, assuming that the standard deviation of random residuals is zero.

Successively, the equilibrium approach has been applied to within-day dynamic assignment models leading to dynamic user equilibrium models with "deterministic" users' choice behaviour [21, 23] and with "stochastic" users' choice behaviour [2, 3, 34]. A different approach is based on the definition of instantaneous "deterministic" dynamic user equilibrium [8, 9, 20, 30, 35, 37], in which it is assumed that equilibrium conditions between path flows and costs are reached at each instant. In this case a continuous-time approach is followed based on optimal control theory (even if some discretisation is necessary to solve the resulting model).

A general formulation is given by

$$\mathbf{F}^* = P(\mathbf{C}^*) \cdot \mathbf{d}, \tag{5.27}$$

$$\mathbf{C}^* = \Gamma[r(\Phi\,[\mathbf{F}^*])], \tag{5.28}$$

or

$$\mathbf{F}^* = P(\Gamma[r(\Phi\,[\mathbf{F}^*])]) \cdot \mathbf{d}. \tag{5.29}$$

It should be stressed that in the within-day dynamic case, only the path flow formulation is generally possible. Again the case of "deterministic" users' behaviour, described in Section 3.2, can be seen as a limit case of this formulation, assuming that the standard deviation of random residuals is zero.

It can easily be recognised that the equilibrium state described by (5.23) or (5.26) for the within-day static case or by (5.29) for the within-day dynamic case is equivalent to a fixed-point attractor of a deterministic dynamic process, described by (5.19), (5.20) or (5.9), respectively, that is, assuming that a QP approach to switching behaviour is followed and conservative filters are adopted.

It is worth noting that for both within-day static and dynamic models, equilibrium states do not depend on parameters of the forecasting filter $Y(-)$ and path switching model $Q(-)$ characterising the day-to-day dynamics of the system.

On the other hand, the equilibrium approach strongly relies on the implicit assumptions of existence, uniqueness and stability of the equilibrium states. Moreover, the length of the transient needed to reach this state is implicitly assumed to be negligible in comparison to the time-lag between two successive modifications of the system in demand and/or supply.

It can be shown that quite mild continuity assumptions for cost functions (and choice functions for "stochastic" user equilibrium) ensure the existence of equilibria (see, among others, [31]). However, the uniqueness relies on somewhat stronger "monotonicity" assumptions on cost functions, and stability still remains a questionable issue [6, 22]. Even in the case of a single equilibrium, if it is not stable, the system will evolve towards a different attractor, unless the starting state is the equilibrium state itself and no fluctuation, however small, occurs. In other words, for realistic values of control parameters the system may not actually evolve toward an equilibrium state, even if it exists and is unique, since the equilibrium stability is greatly affected by the learning and forecasting process and the switching behaviour.

Cascetta and Contarella [17] have analysed in detail a simplified within-day static model, given by learning filter (3.3a) and switching behaviour (3.7), to single out the role of several factors affecting equilibrium stability on a general network (a broad analysis of a simple two-link network is reported in [6]):

$$\mathbf{y}^t = \beta \cdot c(\mathbf{f}^{t-1}) + (1-\beta) \cdot \mathbf{y}^{t-1}, \tag{5.30}$$

$$\mathbf{f}^t = \alpha \cdot \mathbf{A} \cdot P(\mathbf{A}' \cdot \mathbf{y}^t) \cdot \mathbf{d} + (1-\alpha) \cdot \mathbf{f}^{t-1}. \tag{5.31}$$

It is worth noting that using this formulation fixed-point attractors and (stochastic) user equilibrium states coincide, thus the equilibrium stability can be investigated by dealing with the corresponding fixed-point attractor stability.

It was found that the equilibrium stability is greatly influenced by the values of control parameters, and generally any change of parameters increasing the elasticity of the system will shift equilibria towards instability, and generate other types of attractors. In particular, any increase in demand and/or supply leading to an increase in propensity to reconsider the previous day's choice (switching behaviour), and/or an increase in the weight of previous day's experience (learning and forecasting process) and/or an increase of elasticity of choice behaviour, e.g. due to a reduction of perception error or an increase in congestion, will lead equilibrium towards instability.

On the other hand, if the network is not congested or the users' choice behaviour is completely random and thus does not depend on costs, the equilibrium is always stable. The same condition occurs in the limit if the propensity to reconsider previous day's choice in switching behaviour and the weight to yesterday experience in forecasting process tend to zero.

If equilibrium states are searched, apart from any consideration about their stability, they can be found by looking for fixed-point attractors of a simple determinsitic process, as for instance in the within-day static case the model given by (5.30) and (5.31). In this case, t is the index of the algorithm iterations and the behavioural meaning of parameters α and β could be neglected, thus their values could be chosen in order to improve algorithmic efficiency. It is relevant to note that Cascetta and Cantarella [17] showed that in this case small enough values of parameters α and β to assure stable equilibrium always exist.

Algorithms usually adopted for user equilibrium can be cast in the above framework with $\beta = 1$ and α depending on iteration index t in such a way that convergence is assured. From this point of view the method of successive averages [12, 31], often used for computing "stochastic" user equilibrium, uses $\alpha^t = 1/t$. Similarly, the Frank–Wolfe algorithm for "deterministic" user equilibrium [12, 31] uses a variable α^t. Similar algorithms could be, in principle, built up assuming $\alpha = 1$ and β small enough to assure convergence, possibly varying with iteration.

5.3. *Stochastic day-to-day dynamic process models*

Transportation demand, choice fractions and path costs should be generally modelled through random variables, whose values cannot be predicted in advance. This implies that the deterministic process model outlined in Section 5.1 should be seen as an approximation of a stochastic process model, which is closer to the actual system.

In other words, actual values of fractions for a given day cannot be predicted in advance, even if the whole "history" of the system were known, and the evolution of the system among feasible states in successive days should be described as a stochastic process, with properties depending on the hypotheses made on users' behaviour and network configuration. Stochastic day-to-day dynamic models have been recently proposed both for within-day static [11, 15] and within-day dynamic [14, 16] systems. Main results are briefly reported in the following.

In particular, the probability that the path flow vector gets a given value $\mathbf{F}^t$ at day t can be computed, at least theoretically, from choice probabilities. If it is assumed that users are significantly influenced in their choices by at most a limited number (m) of past days (type-b filters), then since path travel times and costs in congested networks depend on link flows, it turns out that the probability that the system is in a given state $\mathbf{F}^t$ at day t depends on the states occupied by the system in m previous days.

Hence assuming that the system state is described as $\{\mathbf{F}^{t-h}: h = 0, 1, \ldots, m\}$, the number of feasible states, i.e. path flow vectors with non-negative components and consistent with demand, is finite if with no loss of generality they are also assumed integer (with any desired precision); thus the resulting stochastic process is an m-dependent Markov chain.

It can be proved that the process admits a unique stationary probability distribution and it is ergodic if, in addition to the limited memory assumption (type-b filter), the following (sufficient) conditions hold [11]:

(i) choice probabilities (but not choice fractions), given the same sequence of costs relative to the previous day, and possibly to the same day, are time-homogeneous, i.e. invariant with respect to a time translation,

(ii) for each pair of different states there is at least one sequence of feasible states, with strictly positive probabilities, from one to the other.

Existence and uniqueness of the stationary distribution ensure that one probability distribution of system states and in particular of path and link flows can be associated to each demand/supply system, independently of the starting configuration. Process ergodicity allows the computation of flow means and moments through the simulation of only one realisation of the process. It is also worth noting that the above properties do not require any hypothesis on the type of link cost functions.

These considerations approximately apply also if a non-finite memory type of filters (type-a) are adopted, since type-a filters can be approximated through type-b filters with any desired precision. More precisely, the system state is generally partially continuous, and the resulting process is an 1-dependent Markov process. It is also worth noting that random events modifying network performances, such as incidents and bad weather conditions, can be included in the proposed framework if their occurrence probability is stable over time.

Obviously it is possible to study transitions between two stationary states of the system. In this case the process ergodicity does not apply and flow moments must be computed over repeated simulations of transients.

The solution of a stochastic process model can be obtained through Monte Carlo techniques, used together with recursive equations. In particular, at day t the fraction of users reconsidering previous day's choice and the path choice fractions are computed through random sampling from the corresponding probabilities. Moreover, if the link costs are assumed random variables, they are computed through random sampling once a probability distribution has been specified.

Relationships between deterministic and stochastic process models are an open research field in the theory of non-linear dynamic systems. In other words, to the authors' knowledge, no general results exist relating dynamic behaviour and attractors of the deterministic process with moments, auto correlation and cross-correlation structure of the stochastic process.

On the other hand, deterministic process models are essentially tools for the analysis of the dynamic structure underlying the system evolution over time, making use of analytical results available in dynamic non-linear system theory. No such general results are available for stochastic process models which, instead, can give a complete statistical description of the temporal evolution of the system.

It can be said that the "variance" of the deterministic process is zero if the system evolves towards a fixed-point (or equilibrium state), while the stochastic process exhibits a variance and a correlation structure even in this case. On the other hand, if the deterministic system evolves towards a periodic or a chaotic attractor, it also has a variance and a correlation structure whose relationship with their stochastic counterparts is not pre-determined. In both cases flow distribution can be expected to exhibit different modes, and the auto correlogram of the stochastic process will show the same structure as the deterministic one but with less extreme values.

From the practical point of view it can be concluded that if O–D demand and capacity values are large enough, deterministic and stochastic descriptions of system evolution should be, according to the law of large numbers, increasingly similar. All these expectations have been confirmed by numerical examples [17], as briefly shown in the next section.

Relationships between equilibrium and stochastic process analysis can be established in a mediate or direct way. In the first case if the deterministic process converges to a fixed-point attractor coincident with an equilibrium state and the system dimensions are large enough, the same can be said for the stochastic process; in this case flow variances should be increasingly small and average steady-state values should converge to equilibrium ones.

The same type of results can be obtained by directly studying relationships between equilibrium and stochastic process average flows as done by Cascetta [11], for the model given by (5.30) and (5.31) assuming $\alpha = 1$ and $\beta = 1$. In this case expected path flows, $E_F[\mathbf{F}]$, and user equilibrium path flows, $\mathbf{F}^* = P(\mathbf{C}(\mathbf{F}^*)) \cdot \mathbf{d}$, coincide exactly only if link cost functions and path choice functions are linear. In general, equality between equilibrium and average path flows is not guaranteed, because average values of non-linear cost functions differ from the values computed with average flows.

6. Numerical examples

In this section some numerical applications of network dynamics, reported elsewhere in the literature are briefly summarised with the aim of demonstrating some of the issues previously reported. The first example, relative to a small test network, is mainly intended to show different types of day-to-day dynamic behaviour, and the influence of control parameters on the relationship between deterministic and stochastic process models [17]. The second example, relative to a small realistic network, aims at showing the possible outcomes of within-day dynamic models, and at the same time the possibility of applying dynamic models for realistic networks [14].

6.1. First example

The test network, with 9 nodes and 12 links, is shown in Fig. 1. Links located in the centre of the network have smaller capacities and their cost-flow functions are more sensitive to congestion, while links on the border have larger free-flow times.

Users are considered homogeneous with respect to trip purpose and available deformation. O–D pairs with demand values and demand values are reported in the following table. All possible paths are considered as available and arranged in sets K_i.

O–D	1–8	1–9	9–8	1–5	5–8	1–4	4–8
demand	60	40	40	10	10	10	10
no. of paths	6	2	2	1	1	1	1

The simple specification reported in Section 5, expressed by (5.30) and (5.31) was adopted, users' path choice was modelled through a pre-fixed probability of reconsidering previous day's choice $\alpha = 0.5$ and a Logit model with a parameter θ common to all O–D pairs. Users' memory was modelled through an exponential smoothing filter, with parameter $\beta = 0.5$.

System evolution was simulated for 360 days, but all statistics were computed over the last 300 days in order to allow the system to reach a steady-state behaviour. User equilibrium (UE) flows referred to in the following were computed through the Method of Successive Averages [31], both deterministic (DP) and stochastic (SP) process day-to-day dynamic models were tested.

The evolution over time of the flows on some links is shown graphically in Figs. 2–4, for the tested dynamic models (DP, SP), together with equilibrium flows (denoted by a continuous line). These results were obtained by assuming a LOGIT parameter $\theta = 0.10$ (corresponding to a "free-time" variation coefficient of perception errors equal to 0.20) and by multiplying all the demand values by a common factor $\mu = 2, 3, 4$, in order to analyse the effects of increasing congestion (the maximum possible value of μ without exceeding any link capacity is 5).

Analysis of the above results shows that with uncongested or moderately congested networks ($\mu = 2, 3$) equilibrium (UE) is stable as is the deterministic process (DP) fixed-point attractor; stochastic process model (SP) exhibit moderate fluctuations around average values close enough to equilibrium flows (denoted by a continuous line).

A higher level of congestion ($\mu = 4$) destabilizes the system: the deterministic process evolves

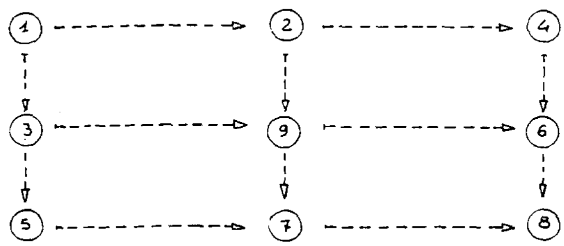

Fig. 1. Small test network.

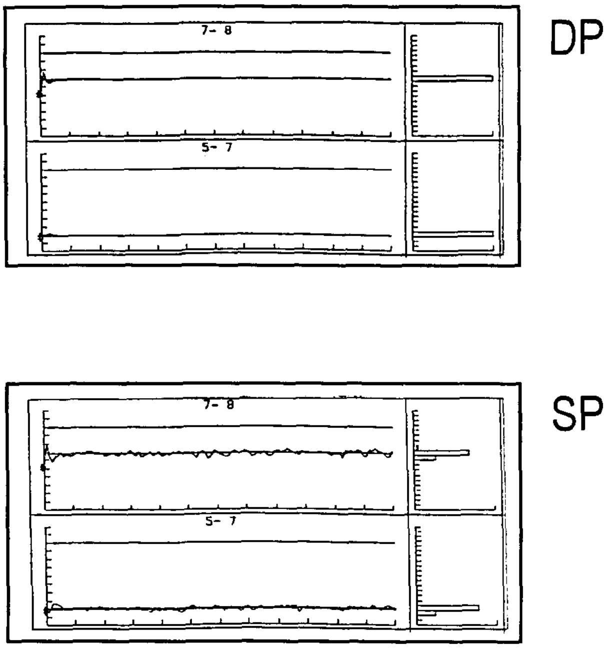

Fig. 2. Evolution over time of link flows $\theta = 0.10$, $\mu = 2$ (fixed-point attractor).

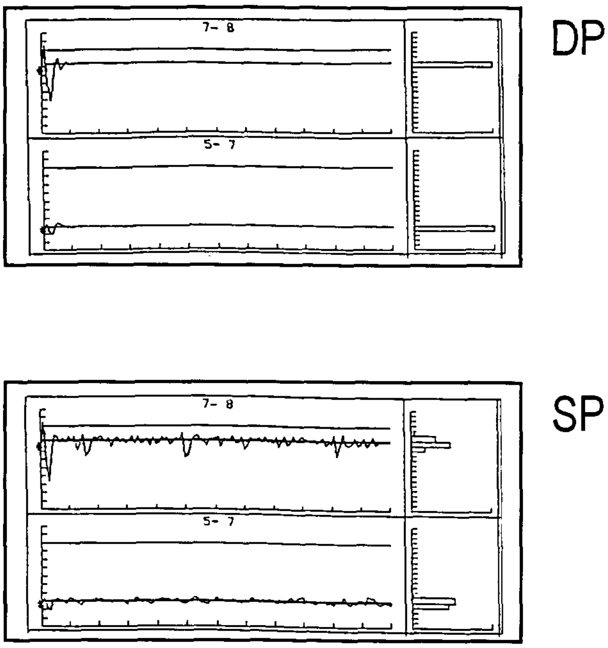

Fig. 3. Evolution over time of link flows $\theta = 0.10$, $\mu = 3$ (fixed-point attractor).

toward a periodic attractor and average flows may significantly differ from those of equilibrium. The stochastic process (SP) shows a pattern quite similar to its deterministic counterpart (DP), even though the random component prevents the system from an exactly periodic behaviour.

The effect of a reduction of perception error was tested by increasing parameter θ from 0.10

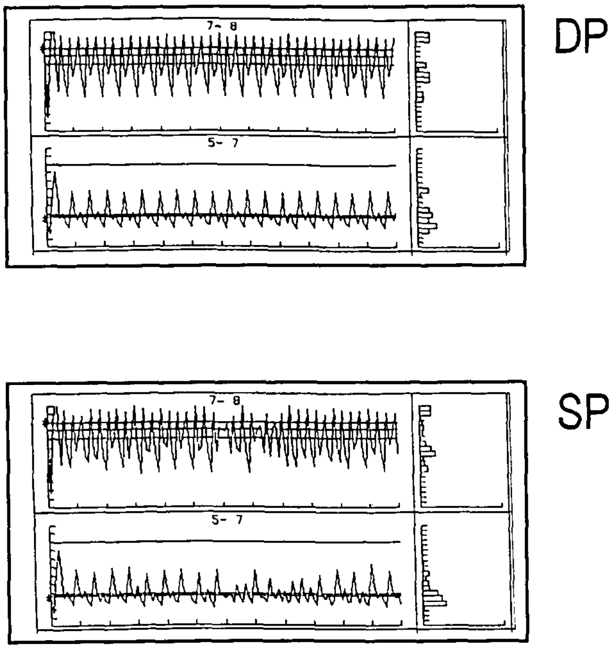

Fig. 4. Evolution over time of link flows $\theta = 0.10$, $\mu = 4$ (fixed-point attractor).

to 0.20 (coefficient of variation reduced approximately from 0.20 to 0.10). Results for level of congestion $\mu = 3$ are reported in Fig. 5. It can be observed that, according to the discussion in the previous section, this variation has similar effect to the increase in congestion, or to any other change increasing the elasticity of the system, leading the equilibrium towards instability.

The presence of different (fractal) attractors can be seen through a slight perturbation of some control parameters. In particular, changing θ from 0.20 to 0.22, other parameters being equal ($\mu = 3$, $\alpha = 0.5$, $\beta = 0.5$), leads the deterministic process to a chaotic attractor, as shown in Fig. 6, with small changes in aggregated statistics.

Moreover, the above examples show that control strategies aimed at the reduction of perception error (from $\theta = 0.10$ to $\theta = 0.20$ or 0.22) may lead the system towards instability.

The relationship between deterministic and stochastic processes is highlighted by the results shown in Figs. 7 and 8, in which both travel demand and link capacity values were multiplied by a common factor equal to 0.1 or 10, for $\theta = 0.10$ and $\mu = 3$. According to the results mentioned in the previous section, the greater the number of users the closer are the two processes.

More considerations and the results of the application of an other stochastic process model are reported in [17].

6.2. Second example

In this section some results relative to an application of a within-day and day-to-day dynamic model to a realistic network are briefly described. The test network refers to the town of Battipaglia with about 30000 inhabitants, in the South of Italy (Fig. 9).

Supply data are relative to the real road network, global O–D demand has been generated through a simple gravity model, and choice behaviour has been modelled by adapting literature models. In particular, to model users' habits it has been assumed that on day t each user confirms the choice made on the previous day $t-1$ if the observed disutility is within a prefixed percentual range of the average predicted one. If this condition is not satisfied the users consider the possibility of changing the previous day's choice. In this case a path/departure time nested

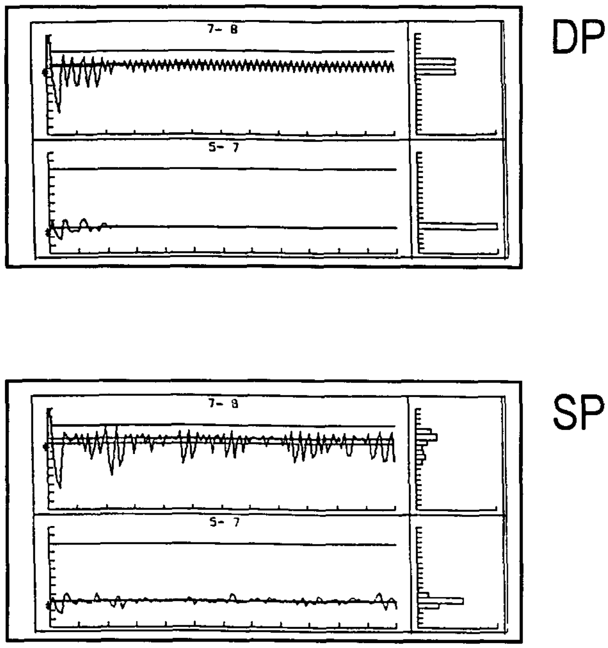

Fig. 5. Evolution over time of link flows $\theta = 0.20$, $\mu = 3$ (periodic attractor).

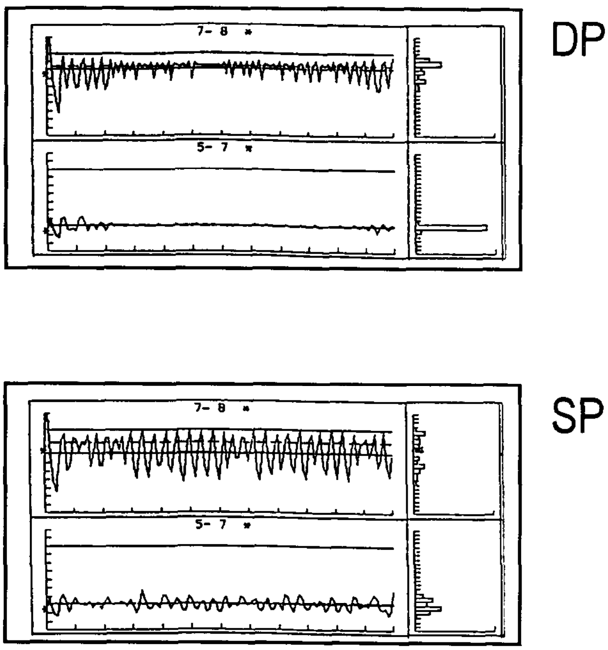

Fig. 6. Evolution over time of link flows $\theta = 0.22$, $\mu = 3$ (fractal attractor).

Logit model is used to simulate the users' departure interval and path choice behaviour. In this case an extra utility for the choice of the same path of previous day $t-1$ (whichever departure interval is chosen), which tries to capture the conservative behaviour of users, is introduced. (For more details see [14].)

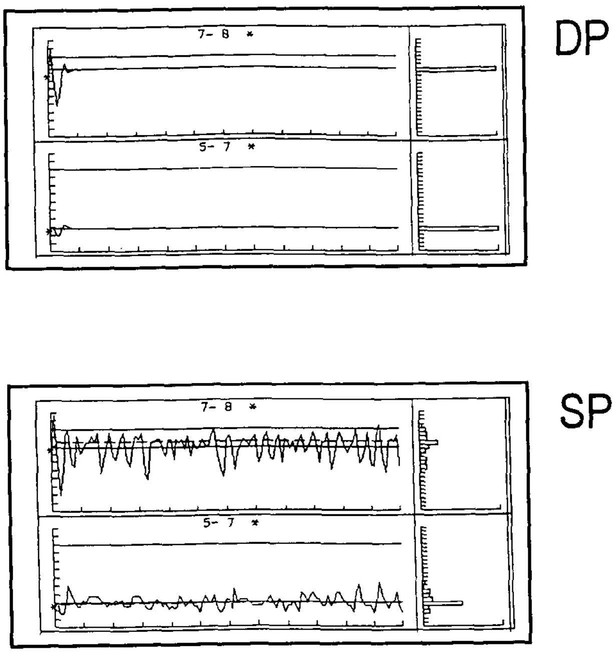

Fig. 7. Evolution over time of link flows (smaller number of users).

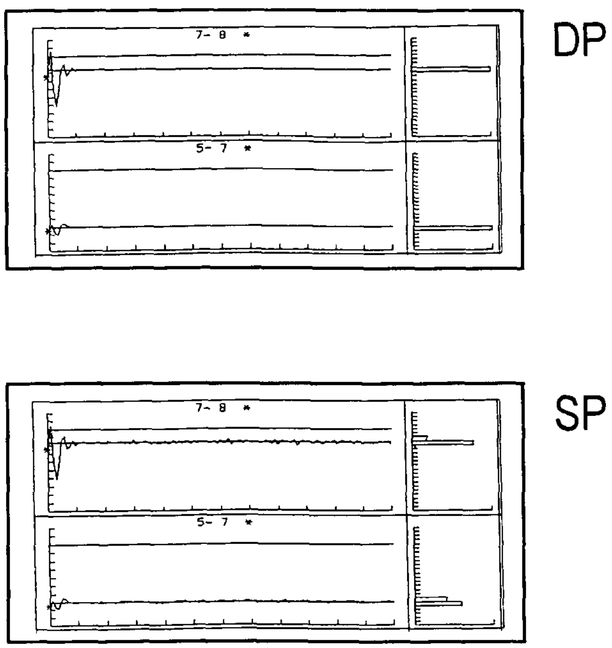

Fig. 8. Evolution over time of link flows (larger number of users).

The simulation period last 60 min (the morning peak hour from 7:30 a.m. to 8:30 a.m.). It has been divided in 12 intervals with a length $T = 5$ min. A user is allowed to leave from the origin in the first 9 intervals (from 7:30 to 8:15); the last three intervals have been included to allow all users to reach their final destination. Systematic disutility has been defined as the sum of

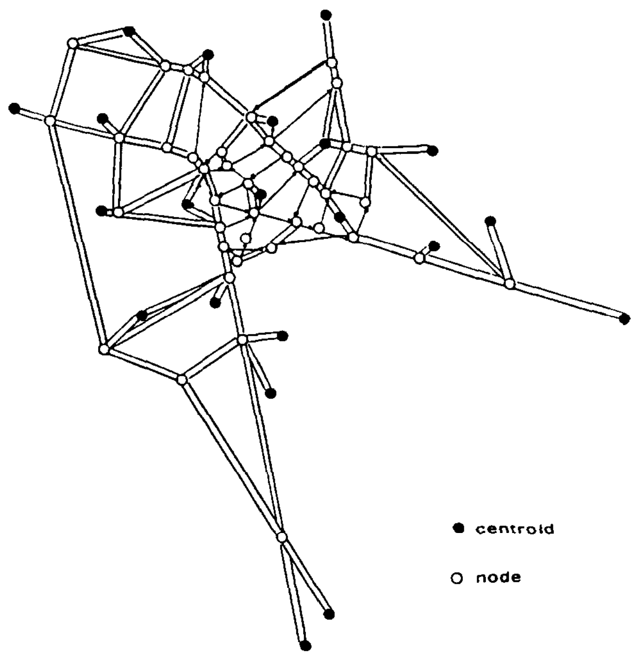

Fig. 9. Battipaglia network.

the travel time and a penalty of early or late arrivals. A common desired arrival time equal to the beginning of interval $h = 7$ has been assumed, with a tolerance band. The average perceived travel time has been computed by using an exponential filter with $\beta = 0.90$.

Stationarity was reached after 30 days. 40 days of simulation were needed to keep arc flow mean estimates within a prefixed sampling error. The average flow patterns in some intervals ($h = 2, 4, 6, 8$) are shown in Figs. 10–13 denoting how congestion spreads on the network.

7. Conclusions

In recent years, the simulation of dynamics in transportation networks have received great attention, since traditional static models seem more suited for long-term planning applications, while short-term planning and operation may require the use of dynamic models, especially if real-time control strategies are involved.

In this paper a theoretical framework for dynamic modelling of transportation networks has been presented, covering both demand and supply modelling. The relevance of day-to-day dynamic models for demand/supply interaction in comparison with more traditional equilibrium approach has been discussed. However, some topics still require research efforts, mainly the modelling of users' behaviour when informative systems are in operations.

Furthermore, data are needed to support empirical validation of the proposed models. The limited empirical evidence available seems to support the existence of a significant day-to-day dynamics in users' route choices. Approximately 30% of commuters' path choices change every day in Turin [18], 50% change at least once a month in Seattle [28]. There are, however, a number of factors not explicitly included in the previously described models tending to smooth (systematic) fluctuations. These factors include path-switching due to non-traffic reasons, such as detours etc. (around 20% of path switches in Turin), topological complexity of the network,

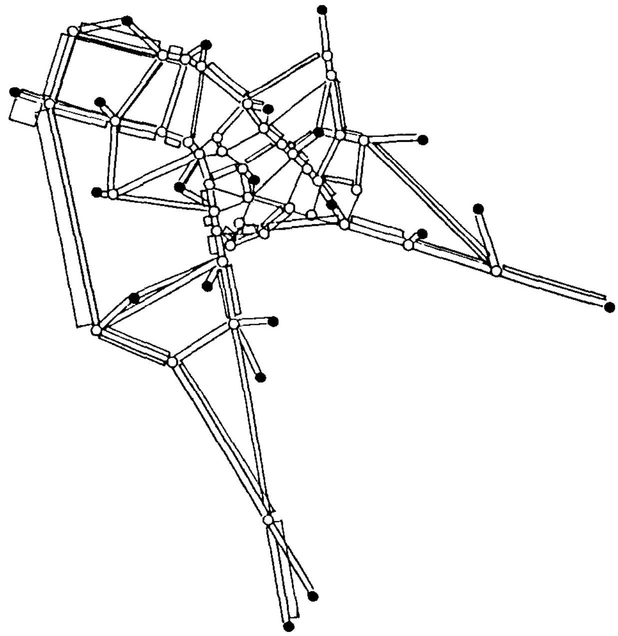

Fig. 10. Link flow pattern during interval $h = 2$.

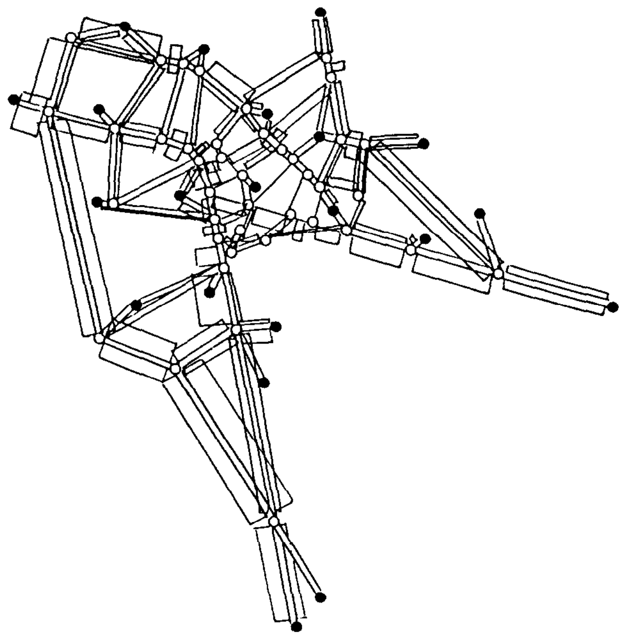

Fig. 11. Link flow pattern during interval $h = 4$.

with several paths comprising the same link with the possibility of mutual compensations, and the presence in path systematic utility of congestion-independent attributes [5].

Finally the specification and validation of algorithms suitable for large scale applications are worthy of research work.

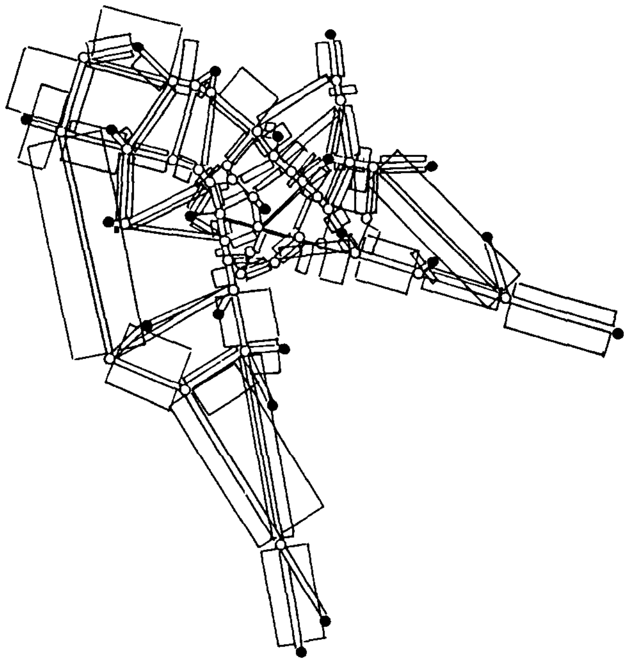

Fig. 12. Link flow pattern during interval $h = 6$.

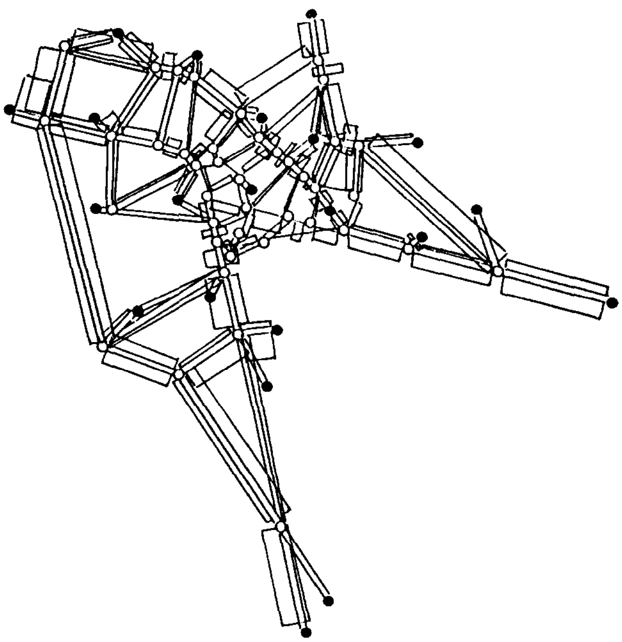

Fig. 13. Link flow pattern during interval $h = 8$.

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